

An Extension of the Chen-Beurling-Helson-Lowdenslager Theorem

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Dedicated to Peter Rosenthal, a great mathematician, a great lawyer, and a great human being

ABSTRACT. Yanni Chen [3] extended the classical Beurling-Helson-Lowdenslager Theorem for Hardy spaces on the unit circle \mathbb{T} defined in terms of continuous gauge norms on L^∞ that dominate $\|\cdot\|_1$. We extend Chen's result to a much larger class of continuous gauge norms. A key ingredient is our result that if α is a continuous normalized gauge norm on L^∞ , then there is a probability measure λ , mutually absolutely continuous with respect to Lebesgue measure on \mathbb{T} , such that $\alpha \geq c\|\cdot\|_{1,\lambda}$ for some $0 < c \leq 1$.

1. Introduction

Let \mathbb{T} be the unit circle, i.e., $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, and let μ be Haar measure (i.e., normalized arc length) on \mathbb{T} . The classical and influential Beurling-Helson-Lowdenslager theorem (see [1],[7]) states that if W is a closed $H^\infty(\mathbb{T}, \mu)$ -invariant subspace (or, equivalently, $zW \subseteq W$) of $L^2(\mathbb{T}, \mu)$, then $W = \varphi H^2$ for some $\varphi \in L^\infty(\mathbb{T}, \mu)$, with $|\varphi| = 1$ a.e. (μ) or $W = \chi_E L^2(\mathbb{T}, \mu)$ for some Borel set $E \subset \mathbb{T}$. If $0 \neq W \subset H^2(\mathbb{T}, \mu)$, then $W = \varphi H^2(\mathbb{T}, \mu)$ for some $\varphi \in H^\infty(\mathbb{T}, \mu)$ with $|\varphi| = 1$ a.e. (μ) . Later, the Beurling's theorem was extended to $L^p(\mathbb{T}, \mu)$ and $H^p(\mathbb{T}, \mu)$ with $1 \leq p \leq \infty$, with the assumption that W is weak*-closed when $p = \infty$ (see [5],[6],[7],[8]). In [3], Yanni Chen extended the Helson-Lowdenslager-Beurling theorem for all continuous $\|\cdot\|_{1,\mu}$ -dominating normalized gauge norms on \mathbb{T} .

In this paper we extend the Helson-Lowdenslager-Beurling theorem for a much larger class of norms. We first extend Chen's results to the case of $c\|\cdot\|_{1,\mu}$ -dominating continuous gauge norms. We then prove that for any continuous gauge norm α , there is a probability measure λ that is mutually absolutely continuous with respect to μ such that α is $c\|\cdot\|_{1,\lambda}$ -dominating. We use this result to extend Chen's theorem. Our extension depends on Radon-Nikodym derivative $d\lambda/d\mu$. In particular, Chen's theorem extends exactly whenever $\log(d\lambda/d\mu) \in L^1(\mathbb{T}, \mu)$.

Key words and phrases. gauge norm, Hardy space, Beurling Theorem.

2. Continuous Gauge Norms on Ω

Suppose (Ω, Σ, ν) is a probability space. A norm α on $L^\infty(\Omega, \nu)$ is a *normalized gauge norm* if

- (1) $\alpha(1) = 1$,
- (2) $\alpha(|f|) = \alpha(f)$ for every $f \in L^\infty(\Omega, \nu)$.

In addition we say α is *continuous* (ν -*continuous*) if

$$\lim_{\nu(E) \rightarrow 0} \alpha(\chi_E) = 0,$$

that is, whenever $\{E_n\}$ is a sequence in Σ and $\nu(E_n) \rightarrow 0$, we have $\alpha(\chi_{E_n}) \rightarrow 0$.

We say that a *normalized gauge norm* α is $c\|\cdot\|_{1,\nu}$ -*dominating* for some $c > 0$ if

$$\alpha(f) \geq c\|f\|_{1,\nu}, \text{ for every } f \in L^\infty(\Omega, \nu).$$

It is easily to see the following fact that

- (1) The common norm $\|\cdot\|_{p,\nu}$ is a α norm for $1 \leq p \leq \infty$.
- (2) If ν and λ are mutually absolutely continuous probability measures, then $L^\infty(\Omega, \nu) = L^\infty(\Omega, \lambda)$ and a normalized gauge norm is ν -continuous if and only if it is λ -continuous.

We can extend the normalized gauge norm α from $L^\infty(\Omega, \nu)$ to the set of all measurable functions, and define α for all measurable functions f on Ω by

$$\alpha(f) = \sup\{\alpha(s) : s \text{ is a simple function}, 0 \leq s \leq |f|\}.$$

It is clear that $\alpha(f) = \alpha(|f|)$ still holds.

Define

$$\mathcal{L}^\alpha(\Omega, \nu) = \{f : f \text{ is a measurable function on } \Omega \text{ with } \alpha(f) < \infty\},$$

$$L^\alpha(\Omega, \nu) = \overline{L^\infty(\Omega, \nu)}^\alpha, \text{ i.e., the } \alpha\text{-closure of } L^\infty(\Omega, \nu) \text{ in } \mathcal{L}^\alpha.$$

Since $L^\infty(\Omega, \nu)$ with the norm α is dense in $L^\alpha(\Omega, \nu)$, they have the same dual spaces. We prove in the next lemma that the normed dual $(L^\alpha(\Omega, \nu), \alpha)^\# = (L^\infty(\Omega, \nu), \alpha)^\#$ can be viewed as a vector subspace of $L^1(\Omega, \nu)$. Suppose $w \in L^1(\Omega, \nu)$, we define the functional $\varphi_w : L^\infty(\Omega, \nu) \rightarrow \mathbb{C}$ by

$$\varphi_w(f) = \int_\Omega f w d\nu.$$

LEMMA 2.1. *Suppose (Ω, Σ, ν) is a probability space and α is a continuous normalized gauge norm on $L^\infty(\Omega, \nu)$. Then*

- (1) *if $\varphi : L^\infty(\Omega, \nu) \rightarrow \mathbb{C}$ is an α -continuous linear functional, then there is a $w \in L^1(\Omega, \nu)$ such that $\varphi = \varphi_w$,*
- (2) *if φ_w is α -continuous on $L^\infty(\Omega, \nu)$, then*
 - (a) $\|w\|_{1,\mu} \leq \|\varphi_w\| = \|\varphi_{|w|}\|$,

(b) given φ in the dual of $L^\alpha(\Omega, \lambda)$, i.e., $\varphi \in (L^\alpha(\Omega, \lambda))^\#$, there exists a $w \in L^1(\Omega, \lambda)$, such that

$$\forall f \in L^\infty(\Omega, \lambda), \varphi(f) = \int_\Omega f w d\lambda \text{ and } w L^\alpha(\Omega, \lambda) \subseteq L^1(\Omega, \lambda).$$

PROOF. (1) If α is continuous, it follows that, whenever $\{E_n\}$ is a disjoint sequence of measurable sets,

$$\lim_{N \rightarrow \infty} \alpha \left(\chi_{\bigcup_{n=1}^\infty E_n} - \sum_{k=1}^N \chi_{E_k} \right) = \lim_{N \rightarrow \infty} \alpha \left(\chi_{\bigcup_{k=N+1}^\infty E_k} \right) = 0,$$

since $\lim_{N \rightarrow \infty} \nu \left(\bigcup_{k=N+1}^\infty E_k \right) = 0$. It follows that

$$\rho(E) = \varphi(\chi_E)$$

defines a measure ρ and $\rho \ll \nu$. It follows that if $w = d\rho/d\nu$, then

$$\begin{aligned} \|w\|_{1,\nu} &= \sup \left\{ \left| \int_\Omega w s d\nu \right| : s \text{ is simple, } \|s\|_\infty \leq 1 \right\} \\ &= \sup \{ |\varphi(s)| : s \text{ simple, } \|s\|_\infty \leq 1 \} \leq \|\varphi\|. \end{aligned}$$

Hence $w \in L^1(\Omega, \nu)$. Also, since, for every $f \in L^\infty(\Omega, \nu)$

$$|\varphi(f)| \leq \|\varphi\| \alpha(f) \leq \|\varphi\| \|f\|_\infty,$$

we see that φ is $\|\cdot\|_\infty$ -continuous on $L^\infty(\Omega, \nu)$, so it follows that $\varphi = \varphi_w$.

(2a) From (1) we will see $\|w\|_{1,\nu} \leq \|\varphi\|$.

(2b) For any measurable set $E \subseteq \Omega$, and for all $\varphi \in (L^\alpha(\Omega, \lambda))^\#$, define $\rho(E) = \varphi(\chi_E)$. we can prove ρ is a measure as in theorem 2.2, and $\rho \ll \lambda$. By Radon-Nikodym theorem, there exists a function $w \in L^1(\lambda)$ such that, for every measurable set $E \subseteq \Omega$, $\varphi(\chi_E) = \rho(E) = \int_\Omega \chi_E w d\lambda$. Thus $\forall f \in L^\infty(\Omega, \lambda)$, $\varphi(f) = \int_\Omega f w d\lambda = \int_\Omega f w g d\mu = \int_\Omega f w |h| d\mu = \int_\Omega f w u h d\mu = \int_\Omega f \tilde{w} h d\mu$, where $\tilde{w} = wu$, $|\tilde{w}| = |w|$, here $\tilde{w} \in L^1(\Omega, \lambda)$ and g, h as in theorem 2.2, so $\tilde{w} h \in L^1(\mu)$. Therefore, $\varphi(f) = \int_\Omega f \tilde{w} h d\mu$ for all $f \in L^\alpha(\Omega, \lambda)$.

Suppose $f \in L^\alpha(\Omega, \lambda)$, $f = u|f|$, $|u| = 1$. $|f| \in L^\alpha(\Omega, \lambda)$. There exists an increasing positive sequence s_n such that $s_n \rightarrow |f|$ a.e. (μ) , thus $u s_n \rightarrow u|f|$ a.e. (μ) . $\forall w \in L^1(\Omega, \lambda)$, $w = v|w|$, where $|v| = 1$, so we have $\bar{v} s_n \rightarrow \bar{v}|f|$ a.e. (μ) , where \bar{v} is the conjugate of v and $\alpha(\bar{v} s_n - \bar{v}|f|) \rightarrow 0$. Thus $\varphi(\bar{v} s_n) \rightarrow \varphi(\bar{v}|f|)$. On the other hand, we also have $\varphi(\bar{v} s_n) = \int_\Omega \bar{v} s_n w d\lambda \rightarrow \int_\Omega \bar{v}|f| w d\lambda = \int_\Omega |f| |w| d\lambda$ by monotone convergence theorem. Thus $\int_\Omega |f| |w| d\lambda = \int_\Omega |f| \bar{v} w d\lambda = \varphi(\bar{v}|f|) < \infty$, therefore $f w \in L^1(\Omega, \lambda)$, i.e., $w L^\alpha(\Omega, \lambda) \subseteq L^1(\Omega, \lambda)$, where $w \in L^1(\Omega, \lambda)$. \square

THEOREM 2.2. Suppose (Ω, Σ, ν) is a probability space, α is a continuous normalized gauge norm on $L^\infty(\Omega, \nu)$ and $\varepsilon > 0$. Then there exists a constant c with $1 - \varepsilon < c \leq 1$ and a probability measure λ on Σ that is mutually absolutely continuous with respect to ν such that α is $c\|\cdot\|_{1,\lambda}$ -dominating.

PROOF. Let $M = \{\nu(h^{-1}((0, \infty))) : h \in L^1(\Omega, \nu), h \geq 0, \varphi_h \text{ is } \alpha\text{-continuous}\}$. It follows from lemma 2.1 that $M \neq \emptyset$. Choose $\{h_n\}$ in $L^1(\Omega, \nu)$ such that $h_n \geq 0$, φ_{h_n} is α -continuous, and such that

$$\nu(h_n^{-1}((0, \infty))) \rightarrow \sup M.$$

Let

$$h_0 = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\|\varphi_{h_n}\|} h_n.$$

Since $\|h_n\|_{1,\nu} \leq \|\varphi_{h_n}\|$, we see that $\|h_0\|_{1,\nu} \leq 1$. Also

$$\varphi_{h_0} = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\|\varphi_{h_n}\|} \varphi_{h_n},$$

so φ_{h_0} is α -bounded and $\|\varphi_{h_0}\| \leq 1$. On the other hand $h_n^{-1}((0, \infty)) \subset h_0^{-1}((0, \infty))$ for $n \geq 1$, so we have

$$\nu(h_0^{-1}((0, \infty))) = \sup M.$$

Let $E = \Omega \setminus h_0^{-1}((0, \infty))$ and assume, via contradiction, that $\nu(E) > 0$. Then $\alpha(\chi_E) > 0$. Hence, by the Hahn-Banach theorem, there is a $g \in L^1(\Omega, \nu)$ such that $\|\varphi_g\| = 1$ and

$$\alpha(\chi_E) = \varphi_g(\chi_E) = \int_{\Omega} g \chi_E d\nu = \varphi_{g\chi_E}(\chi_E) \leq \varphi_{|g|\chi_E}(\chi_E).$$

It follows that $\nu((|g|\chi_E)^{-1}(0, \infty)) = \eta > 0$, and that if $h_1 = h_0 + |g|\chi_E$, then

$$\sup M \geq \nu(h_1^{-1}((0, \infty))) = \nu(h_0^{-1}((0, \infty))) + \eta = \sup M + \eta.$$

This contradiction shows that $\nu(E) = 0$, so we can assume that $h_0(\omega) > 0$ a.e. (ν) . By replacing h_0 with $h_0 / \int_{\Omega} h_0 d\nu$, we can assume that $\int_{\Omega} h_0 d\nu = 1$.

If we define a probability measure $\lambda : \Sigma \rightarrow [0, 1]$ by

$$\lambda(E) = \int_E h_0 d\nu,$$

then λ is a measure, $\lambda \ll \nu$ and $\nu \ll \lambda$ since $0 < h_0$ a.e. (ν) . Also, we have for every $f \in L^{\infty}(\Omega, \nu)$,

$$\|f\|_{1,\lambda} = \int_{\Omega} |f| d\lambda = \int_{\Omega} |f| h_0 d\nu = \varphi_{h_0}(|f|) \leq \|\varphi_{h_0}\| \alpha(f).$$

Since $\varphi_{h_0}(1) = 1$, we know $\|\varphi_{h_0}\| \geq 1$. Hence, $0 < c_0 = 1/\|\varphi_{h_0}\| \leq 1$, and we see that α is $c_0\|\cdot\|_{1,\lambda}$ -dominating on E . If we apply the Hahn-Banach theorem as above with $E = \Omega$, we can find a nonnegative function $k \in L^1(\Omega, \nu)$ such that

$$\|\varphi_k\| = 1 = \alpha(1) = \varphi_k(1) = \int_{\Omega} k d\nu.$$

For $0 < t < 1$ let $h_t = (1-t)k + th_0$. Then $\varphi_{h_t} = (1-t)\varphi_k + t\varphi_{h_0}$. Thus

$$\lim_{t \rightarrow 0^+} \|\varphi_{h_t}\| = \|\varphi_k\| = 1.$$

Choose t so that $\|\varphi_{h_t}\| < 1/(1 - \varepsilon)$, so $1 - \varepsilon < c = 1/\|\varphi_{h_t}\| \leq 1$. If we define a probability measure $\lambda_t : \Sigma \rightarrow [0, 1]$ by

$$\lambda_t(E) = \int_E h_t d\nu,$$

we see that $\lambda_t \ll \mu\nu$ and since $h_t \geq th_0 > 0$, we see $\nu \ll \lambda_t$. As above we see, for every $f \in L^\infty(\Omega, \mu)$ we have

$$c\|f\|_{1, \lambda_t} \leq \frac{1}{\|\varphi_{h_t}\|} \int_\Omega |f| h_t d\nu = \frac{1}{\|\varphi_{h_t}\|} \varphi_{h_t}(|f|) \leq \alpha(f).$$

Therefore, α is $c\|\cdot\|_{1, \lambda_t}$ -dominating on Ω . \square

If we take $\Omega = \mathbb{T}$, theorem 2.2 holds for the probability space $(\Omega, \nu) = (\mathbb{T}, \mu)$. The L^p -version of the Helson-Lowdenslager theorem also holds, in a sense, on the circle \mathbb{T} when μ is replaced with a mutually absolutely continuous probability measure λ . Here the role of $H^p(\mathbb{T}, \lambda)$ is replaced with $(1/g^{\frac{1}{p}})H^p(\mathbb{T}, \mu)$. This result is well-known, we include a proof for completeness as the following corollary.

COROLLARY 2.3. *Suppose λ is a probability measure on \mathbb{T} and $\mu \ll \lambda$ and $\lambda \ll \mu$. Let $g = d\lambda/d\mu$ and suppose $1 \leq p < \infty$. Suppose W is a closed subspace of $L^p(\mathbb{T}, \lambda)$, and $zW \subset W$. Then $g^{\frac{1}{p}}W = \chi_E L^1(\mathbb{T}, \mu)$ for some Borel subset E of \mathbb{T} or $g^{\frac{1}{p}}W = \varphi H^p(\mathbb{T}, \mu)$ for some unimodular function φ .*

PROOF. Define $U : L^p(\mathbb{T}, \lambda) \longrightarrow L^p(\mathbb{T}, \mu)$ by $Uf = fg^{\frac{1}{p}}$, for $f \in L^p(\mathbb{T}, \lambda)$. Clearly U is a surjective isometry, since

$$\|Uf\|_{p, \mu}^p = \int_{\mathbb{T}} |fg^{\frac{1}{p}}|^p d\mu = \int_{\mathbb{T}} |f|^p g d\mu = \int_{\mathbb{T}} |f|^p d\lambda = \|f\|_{p, \lambda}^p.$$

Define

$$M_{z, \mu} : L^p(\mathbb{T}, \mu) \longrightarrow L^p(\mathbb{T}, \mu) \text{ by } M_{z, \mu}f = zf \text{ and } M_{z, \lambda} : L^p(\mathbb{T}, \lambda) \longrightarrow L^p(\mathbb{T}, \lambda) \text{ by } M_{z, \lambda}f = zf.$$

Then

$$UM_{z, \lambda}f = U(zf) = g^{\frac{1}{p}}zf = zg^{\frac{1}{p}}f = M_{z, \mu}g^{\frac{1}{p}}f = M_{z, \mu}Uf,$$

so $UM_{z, \lambda} = M_{z, \mu}U$. It follows that W is a closed z -invariant subspace of $L^p(\mathbb{T}, \lambda)$ if and only if $g^{\frac{1}{p}}W = U(W)$ is a z -invariant closed linear subspace of $L^p(\mathbb{T}, \mu)$. The conclusion now follows from the classical Beurling theorem for $L^p(\mathbb{T}, \mu)$. \square

3. Continuous Gauge Norms on the Unit Circle

Suppose α is a continuous normalized gauge norm on $L^\infty(\mathbb{T}, \mu)$, suppose that $c > 0$ and λ is a probability measure on \mathbb{T} such that $\lambda \ll \mu$ and $\mu \ll \lambda$ and such that α is $c\|\cdot\|_{1, \lambda}$ -dominating. We let $g = d\lambda/d\mu$ and $g > 0$. We consider two cases

- (1) $\int |\log g| d\mu < \infty$,
- (2) $\int |\log g| d\mu = \infty$.

We define $L^p(\mathbb{T}, \lambda)$ to be the $\|\cdot\|_{p,\lambda}$ -closure of $L^\infty(\mathbb{T}, \lambda)$ and define $H^p(\mathbb{T}, \lambda)$ to be $\|\cdot\|_{p,\lambda}$ -closure of the polynomials for $1 \leq p < \infty$. Denote $L^\infty(\mathbb{T}, \mu) = L^\infty(\mu)$, $L^p(\mathbb{T}, \mu) = L^p(\mu)$ and $H^p(\mathbb{T}, \mu) = H^p(\mu)$.

LEMMA 3.1. *The following are true:*

- (1) $\int |\log g| d\mu < \infty \Leftrightarrow$ *there is an outer function $h \in H^1(\mu)$ with $|h| = g$,*
- (2) $\int |\log g| d\mu = \infty \Leftrightarrow H^1(\lambda) = L^1(\lambda)$.

PROOF. Clearly $H^1(\lambda)$ is a closed z -invariant subspace of $L^1(\lambda)$. Thus, by corollary 2.3, either $gH^1(\lambda) = \varphi H^1(\mu)$ for some unimodular φ or $gH^1(\lambda) = \chi_E L^1(\mu)$ for some Borel set $E \subset \mathbb{T}$.

For (1), if $gH^1(\lambda) = \varphi H^1(\mu)$ for some unimodular φ , and $0 < g \in gH^1(\lambda)$, then $0 \neq \overline{\varphi}g \in H^1(\mu)$ which implies $\log g = \log |\overline{\varphi}g| \in L^1(\mu)$. It is a standard fact that if $g > 0$ and $\log g$ are in $L^1(\mu)$, then there exists an outer function $h \in H^1(\mu)$ with the same modulus as g , (i.e., $|h| = g$). Therefore, (1) is proved by lemma 3.2 in [3].

For (2), Since $gH^1(\lambda) = \varphi H^1(\mu)$ if and only if $\int |\log g| d\mu < \infty$. Suppose $\int |\log g| d\mu = \infty$, then $gH^1(\lambda) = \chi_E L^1(\mu)$. We have $g = \chi_E f$ for some $f \in L^1(\mu)$, which implies $\chi_E = 1$ since $g > 0$. Thus $gH^1(\lambda) = L^1(\mu) = gL^1(\mu)$, which implies $H^1(\lambda) = L^1(\lambda)$. Conversely, if $H^1(\lambda) = L^1(\lambda)$, then $gH^1(\lambda) = gL^1(\lambda) = L^1(\mu) = \chi_{\mathbb{T}} L^1(\mu)$, which means $gH^1(\lambda) \neq \varphi H^1(\mu)$, i.e., $\int |\log g| d\mu = \infty$. \square

There is an important characterization of outer functions in $H^1(\mu)$.

LEMMA 3.2. *A function f is an outer function in $H^1(\mu)$ if and only there is a real harmonic function u with harmonic conjugate \overline{u} such that*

- (1) $u \in L^1(\mu)$,
- (2) $f = e^{u+i\overline{u}}$,
- (3) $f \in L^1(\mu)$.

Through the remainder of following sections we assume

- (1) α is a continuous normalized gauge norm on $L^\infty(\mu)$.
- (2) and that $c > 0$ and λ is a probability measure on \mathbb{T} such that $\lambda \ll \mu$ and $\mu \ll \lambda$ and such that α is $c\|\cdot\|_{1,\lambda}$ dominating.
- (3) $h \in H^1(\mu)$ is an outer function, η is unimodular and $\bar{\eta}h = g = d\lambda/d\mu$.

Since λ and μ are mutually absolutely continuous we have $L^\infty(\mu) = L^\infty(\lambda)$, $L^\alpha(\mu) = L^\alpha(\lambda)$ and $H^\alpha(\mu) = H^\alpha(\lambda)$, we will use L^∞ to denote $L^\infty(\mu)$ and $L^\infty(\lambda)$, use L^α to denote $L^\alpha(\mu)$ and $L^\alpha(\lambda)$, use H^α to denote $H^\alpha(\mu)$ and $H^\alpha(\lambda)$. It follows that $L^\alpha, L^\infty, H^\alpha$ do not depend on λ or μ . However, this notation slightly conflicts with the classical notation for $L^1(\mu) = L^{\|\cdot\|_{1,\mu}}$ or $H^1(\mu) = H^{\|\cdot\|_{1,\mu}}$, so we will add the measure to the notation when we are talking about L^p or H^p .

THEOREM 3.3. *We have $hL^1(\lambda) = L^1(\mu)$ and $hH^1(\lambda) = H^1(\mu)$.*

PROOF. We know from our assumption (3) that $hL^1(\lambda) = g\eta L^1(\lambda) = gL^1(\lambda) = L^1(\mu)$. By lemma 3.1(1), we have $gH^1(\lambda) = \eta H^1(\mu)$, so

$$hH^1(\lambda) = \eta gH^1(\lambda) = \eta\eta H^1(\mu) = H^1(\mu).$$

□

COROLLARY 3.4. $gH^1(\lambda) = \gamma H^1(\mu)$ for some unimodular $\gamma \Leftrightarrow \int_{\mathbb{T}} |\log g| d\mu < \infty$.

PROOF. Assume $gH^1(\lambda) = \gamma H^1(\mu)$, Since $1 \in H^1(\lambda)$, $g \in gH^1(\lambda)$, $\exists \phi \in H^1(\mu)$ such that $g = \gamma\phi$. Since $\phi \in H^1(\mu)$, $\phi = \psi h$, where ψ is an inner function and h is an outer function. Thus, $\int_{\mathbb{T}} |\log g| d\mu = \int_{\mathbb{T}} \log |g| d\mu = \int_{\mathbb{T}} \log |h| d\mu < \infty$, since h is an outer function.

Assume $\int_{\mathbb{T}} |\log g| d\mu < \infty$, g and $\log g \in L^1(\mu)$, $g > 0$. Thus there exists an outer function $h \in H^1(\mu)$, such that $|h| = |g| = g$, $|h| = \phi h$, $|\phi| = 1$, $g = \eta h$. Define $V : L^1(\lambda) \rightarrow L^1(\mu)$ by $Vf = hf$, as in the theorem 3.3, we have $hH^1(\lambda) = H^1(\mu)$, so $gH^1(\lambda) = \eta hH^1(\lambda) = \eta H^1(\mu)$. Let $\gamma = \eta$, then $gH^1(\lambda) = \gamma H^1(\mu)$. □

We now get a Helson-Lowdenslager theorem when $\alpha = \|\cdot\|_{p,\lambda}$ and $\log g \in L^1(\mu)$.

COROLLARY 3.5. Suppose $1 \leq p < \infty$. If W is a closed subspace of $L^p(\lambda)$ and $zW \subseteq W$, then either $W = \gamma H^p(\lambda)$ for some unimodular function γ , or $W = \chi_E L^p(\lambda)$ for some Borel subset E of \mathbb{T} .

The following theorem shows the relation between H^α , $H^1(\lambda)$ and L^α . This result parallels a result of Y Chen [3], which is a key ingredient in her proof of her general Beurling theorem. However, her result was for $H^1(\mu)$ instead of $H^1(\lambda)$.

THEOREM 3.6. $H^\alpha = H^1(\lambda) \cap L^\alpha$.

PROOF. Since α is continuous $c\|\cdot\|_{1,\lambda}$ -dominating, α -convergence implies $\|\cdot\|_{1,\lambda}$ -convergence, thus

$$H^\alpha = \overline{H^\infty}^\alpha \subseteq \overline{H^\infty}^{\|\cdot\|_{1,\lambda}} = H^1(\lambda).$$

Also,

$$H^\alpha = \overline{H^\infty(\lambda)}^\alpha \subset \overline{L^\infty}^\alpha = L^\alpha.$$

Thus $H^\alpha \subseteq H^1(\lambda) \cap L^\alpha$.

Since α -convergence implies $\|\cdot\|_{1,\lambda}$ -convergence, $H^1(\lambda) \cap L^\alpha$ is an α -closed subspace of L^α . Suppose $\varphi \in (L^\alpha)^\#$ such that $\varphi|_{H^\infty} = 0$. It follows from lemma 2.1 that there is a $w \in L^1(\lambda)$ such that $wL^\alpha \subset L^1(\lambda)$ and such that, for every $f \in L^\alpha$,

$$\varphi(f) = \int f \bar{\eta} w d\lambda = \int f w h d\mu.$$

Since $wL^\alpha \subset L^1(\lambda)$, we know that $whL^\alpha \subset L^1(\mu)$. Since $\varphi|_{H^\infty} = 0$, we have

$$\int_{\mathbb{T}} z^n h w d\mu = \varphi(z^n) = 0$$

for every integer $n \geq 0$. Thus $hw \in H_0^1(\mu)$.

Now suppose $f \in H^1(\lambda) \cap L^\alpha$. Then $hf \in H^1(\mu)$. We know that every function in $H^1(\mu)$ has a unique inner-outer factorization. Thus we can write

$$hf = \gamma_1 h_1$$

with γ_1 inner and h_1 outer. Moreover, since $hw \in H_0^1(\mu)$, we can write

$$(hw)(z) = z\gamma_2(z)h_2(z)$$

with γ_2 inner and h_2 outer. By lemma 3.2, we can find real harmonic functions $u, u_1, u_2 \in L^1(\mu)$ such that

$$h = e^{u+i\bar{u}}, h_1 = e^{u_1+i\bar{u}_1}, \text{ and } h_2 = e^{u_2+i\bar{u}_2}.$$

Thus

$$hfw = hfhw/h = z\gamma_1\gamma_2 e^{(u_1+u_2-u)+i(\bar{u}_1+\bar{u}_2-\bar{u})} \in H^1(\mu).$$

It follows from lemma 3.2 that

$$\varphi(f) = \int_{\mathbb{T}} hfw d\mu = (hfw)(0) = 0.$$

Hence every continuous linear functional on L^α that annihilates H^α also annihilates $H^1(\lambda) \cap L^\alpha$. It follows from the Hahn-Banach theorem that $H^1(\lambda) \cap L^\alpha \subset H^\alpha$. \square

The following result is a factorization theorem for L^α .

THEOREM 3.7. *If $k \in L^\infty$, $k^{-1} \in L^\alpha$, then there is a unimodular function $u \in L^\infty$ and an outer function $s \in H^\infty$ such that $k = us$ and $s^{-1} \in H^\alpha$.*

PROOF. Recall that an outer function is uniquely determined by its absolute boundary values, which are necessarily absolutely log integrable. Since $k^{-1} \in L^\alpha \subseteq L^1(\lambda)$, we know that $\|k\|_\infty > 0$. Thus $\log |k| \leq \log \|k\|_\infty \in \mathbb{R}$. Moreover, $k^{-1} \in L^\alpha \subseteq L^1(\lambda)$ implies $hk^{-1} \in L^1(\mu)$, so

$$\log |h| - \log |k| = \log (|hk^{-1}|) \leq |hk^{-1}|.$$

Hence

$$\log |h| - |hk^{-1}| \leq \log |k| \leq \log \|k\|_\infty,$$

and since $\log |h|$, $|hk^{-1}|$ and $\log \|k\|_\infty$ are in $L^1(\mu)$, we see that $\log |k| \in L^1(\mu)$. Therefore, by the first statement of lemma 3.1, there is an outer function $s \in H^1(\mu)$ such that $|s| = |k|$. It follows that $s \in H^\infty$. Hence there is a unimodular function u such that $k = us$.

We also know that

$$|\log |hk^{-1}|| = |\log (|h|) - \log |k|| \leq |\log (|h|)| + |\log |k|| \in L^1(\mu),$$

so there exists an outer function $f \in H^1(\mu)$ such that $|k^{-1}h| = |f|$. Thus sf is outer in $H^1(\mu)$ and $|h| = |sf|$, so $h = e^{it}sf$ for some real number t . Since $H^1(\mu) = hH^1(\lambda)$, we see that there exists a function $f_1 \in H^1(\lambda)$ such that $hf_1 = f = h(e^{-it}s^{-1})$. It follows that $s^{-1} = e^{it}f_1 \in H^1(\lambda)$. Also, $|s^{-1}| = |k^{-1}|$, so $s^{-1} \in L^\alpha$. It follows from Theorem 3.6 that $s^{-1} \in H^1(\lambda) \cap L^\alpha = H^\alpha$. \square

LEMMA 3.8. *If M is a closed subspace of L^α and $zM \subseteq M$, then $H^\infty M \subseteq M$.*

PROOF. Suppose $\varphi \in (L^\alpha)^\#$ and $\varphi|_M = 0$. It follows from lemma 2.1 that there is a $w \in L^1(\lambda)$ such that $wL^\alpha \subset L^1(\lambda)$ such that, for every $f \in L^\alpha$

$$\varphi(f) = \int_{\mathbb{T}} f w \bar{\eta} d\lambda = \int_T f w h d\mu.$$

Suppose $f \in M$. Then, for every integer $n \geq 0$, we have $z^n f \in M$, so

$$0 = \int_{\mathbb{T}} z^n f w h d\mu.$$

Since $f w h \in hL^1(\lambda) = L^1(\mu)$, it follows that $f w h \in H_0^1(\mu)$. Thus if $k \in H^\infty$, we have

$$0 = \int_{\mathbb{T}} k f w h d\mu = \varphi(kf).$$

Hence every $\varphi \in (L^\alpha)^\#$ that annihilates M must annihilate $H^\infty M$. It follows from the Hahn-Banach theorem that $H^\infty M \subset M$. \square

We let $\mathbb{B} = \{f \in L^\infty : \|f\|_\infty \leq 1\}$ denote the closed unit ball in $L^\infty(\lambda)$.

LEMMA 3.9. *Let α be a continuous norm on $L^\infty(\lambda)$, then*

- (1) *The α -topology, the $\|\cdot\|_{2,\lambda}$ -topology, and the topology of convergence in λ -measure coincide on \mathbb{B} ,*
- (2) *$\mathbb{B} = \{f \in L^\infty(\lambda) : \|f\|_\infty \leq 1\}$ is α -closed.*

PROOF. For (1), since α is $c\|\cdot\|_{1,\lambda}$ -dominating, α -convergence implies $\|\cdot\|_{1,\lambda}$ -convergence, and $\|\cdot\|_{1,\lambda}$ -convergence implies convergence in measure. Suppose $\{f_n\}$ is a sequence in \mathbb{B} , $f_n \rightarrow f$ in measure and $\varepsilon > 0$. If $E_n = \{z \in \mathbb{T} : |f(z) - f_n(z)| \geq \frac{\varepsilon}{2}\}$, then $\lim_{n \rightarrow \infty} (E_n) = 0$. Since α is continuous, we have $\lim_{n \rightarrow \infty} \alpha(\chi_{E_n}) = 0$, which implies that

$$\begin{aligned} \alpha(f_n - f) &= \alpha((f - f_n)\chi_{E_n} + (f - f_n)\chi_{\mathbb{T} \setminus E_n}) \\ &\leq \alpha((f - f_n)\chi_{E_n}) + \alpha((f - f_n)\chi_{\mathbb{T} \setminus E_n}) \\ &< \alpha(|f - f_n|\chi_{E_n}) + \frac{\varepsilon}{2} \leq \|f - f_n\|_\infty \alpha(\chi_{E_n}) + \frac{\varepsilon}{2} \\ &\leq 2\alpha(\chi_{E_n}) + \frac{\varepsilon}{2}. \end{aligned}$$

Hence $\alpha(f_n - f) \rightarrow 0$ as $n \rightarrow \infty$. Therefore α -convergence is equivalent to convergence in measure on \mathbb{B} . Since α was arbitrary, letting $\alpha = \|\cdot\|_{2,\lambda}$, we see that $\|\cdot\|_{2,\lambda}$ -convergence is also equivalent to convergence in measure. Therefore, the α -topology and the $\|\cdot\|_{2,\lambda}$ -topology coincide on \mathbb{B} .

For (2), suppose $\{f_n\}$ is a sequence in \mathbb{B} , $f \in L^\alpha$ and $\alpha(f_n - f) \rightarrow 0$. Since $\|f\|_{1,\lambda} \leq \frac{1}{c}\alpha(f)$, it follows that $\|f_n - f\|_{1,\lambda} \rightarrow 0$, which implies that $f_n \rightarrow f$ in λ -measure. Then there is a subsequence f_{n_k} such that $f_{n_k} \rightarrow f$ a.e. (λ). Hence $f \in \mathbb{B}$. \square

The following theorem and its corollary relate the closed invariant subspaces of L^α to the weak*-closed invariant subspaces of L^∞ .

THEOREM 3.10. *Let W be an α -closed linear subspace of L^α and M be a weak*-closed linear subspace of $L^\infty(\lambda)$ such that $zM \subseteq M$ and $zW \subseteq W$. Then*

- (1) $M = \overline{M}^\alpha \cap L^\infty(\lambda)$,
- (2) $W \cap L^\infty(\lambda)$ is weak*-closed in $L^\infty(\lambda)$,
- (3) $W = \overline{W \cap L^\infty(\lambda)}^\alpha$.

PROOF. For (1), it is clear that $M \subset \overline{M}^\alpha \cap L^\infty(\lambda)$. Assume, via contradiction, that $w \in \overline{M}^\alpha \cap L^\infty(\lambda)$ and $w \notin M$. Since M is weak*-closed, there is an $F \in L^1(\lambda)$ such that $\int_{\mathbb{T}} Fwd\lambda \neq 0$, but $\int_{\mathbb{T}} Frd\lambda = 0$ for every $r \in M$. Since $k = \frac{1}{|F|+1} \in L^\infty(\lambda)$, $k^{-1} \in L^1(\lambda)$, it follows from theorem 3.7, that there is a $s \in H^\infty(\lambda)$, $s^{-1} \in H^1(\lambda)$ and a unimodular function u such that $k = us$. Choose a sequence $\{s_n\}$ in $H^\infty(\lambda)$ such that $\|s_n - s^{-1}\|_{1,\lambda} \rightarrow 0$. Since $sF = \overline{u}kF = \overline{u}\frac{F}{|F|+1} \in L^\infty(\lambda)$, we can conclude that $\|s_nsF - F\|_{1,\lambda} = \|s_nsF - s^{-1}sF\|_{1,\lambda} \leq \|s_n - s^{-1}\|_{1,\lambda}\|sF\|_\infty \rightarrow 0$. For each $n \in \mathbb{N}$. For every $r \in M$, from lemma 3.8, we know that $s_nsr \in H^\infty(\lambda)M \subset M$. Hence

$$\int_{\mathbb{T}} rs_nsf d\lambda = \int_{\mathbb{T}} s_nsrF d\lambda = 0, \forall r \in M.$$

Suppose $r \in \overline{M}^\alpha$. Then there is a sequence r_m in M such that $\alpha(r_m - r) \rightarrow 0$ as $m \rightarrow \infty$. For each $n \in \mathbb{N}$, it follows from $s_nsf \in H^\infty(\lambda)L^\infty(\lambda)$ that

$$\begin{aligned} \left| \int_{\mathbb{T}} rs_nsf d\lambda - \int_{\mathbb{T}} r_ms_nsf d\lambda \right| &\leq \int_{\mathbb{T}} |(r - r_m)s_nsf| d\lambda \\ &\leq \|s_nsf\|_\infty \int_{\mathbb{T}} |r - r_m| d\lambda = \|s_nsf\|_\infty \|r - r_m\|_{1,\lambda} \\ &\leq \|s_nsf\|_\infty \alpha(r - r_m) \rightarrow 0. \\ \int_{\mathbb{T}} rs_nsf d\lambda &= \lim_{m \rightarrow \infty} \int_{\mathbb{T}} r_ms_nsf d\lambda = 0, \forall r \in \overline{M}^\alpha. \end{aligned}$$

In particular, $w \in \overline{M}^\alpha \cap L^\infty(\lambda)$ implies that

$$\int_{\mathbb{T}} s_nsfwd\lambda = \int_{\mathbb{T}} ws_nsf d\lambda = 0.$$

Hence,

$$\begin{aligned} 0 \neq \left| \int_{\mathbb{T}} Fwd\lambda \right| &\leq \lim_{n \rightarrow \infty} \left| \int_{\mathbb{T}} Fw - s_nsfwd\lambda \right| + \lim_{n \rightarrow \infty} \left| \int_{\mathbb{T}} s_nsfwd\lambda \right| \\ &\leq \lim_{n \rightarrow \infty} \|F - s_nsf\|_{1,\lambda} \|w\|_\infty + 0 = 0. \end{aligned}$$

We get a contradiction. Hence $M = \overline{M}^\alpha \cap L^\infty(\lambda)$.

For (2), to prove $W \cap L^\infty(\lambda)$ is weak*-closed in $L^\infty(\lambda)$, using the Krein-Smulian theorem, we only need to show that $W \cap L^\infty(\lambda) \cap \mathbb{B}$, i.e., $W \cap \mathbb{B}$, is weak*-closed. By lemma 3.9, $W \cap \mathbb{B}$ is α -closed. Since α is $c\|\cdot\|_{1,\lambda}$ -dominating, it follows from the lemma 3.9, $W \cap \mathbb{B}$ is $\|\cdot\|_{2,\lambda}$ closed.

The fact that $W \cap \mathbb{B}$ is convex implies $W \cap \mathbb{B}$ is closed in the weak topology on $L^2(\lambda)$. If $\{f_\lambda\}$ is a net in $W \cap \mathbb{B}$ and $f_\lambda \rightarrow f$ weak* in $L^\infty(\lambda)$, then, for every $w \in L^1(\lambda)$, $\int_{\mathbb{T}} (f_\lambda - f)w d\lambda \rightarrow 0$. Since $L^2(\lambda) \subset L^1(\lambda)$, $f_\lambda \rightarrow f$ weakly in $L^2(\lambda)$, so $f \in W \cap \mathbb{B}$. Hence $W \cap \mathbb{B}$ is weak*-closed in $L^\infty(\lambda)$.

For (3), since W is α -closed in L^α , it is clear that $W \supset \overline{W \cap L^\infty(\lambda)}^\alpha$, suppose $f \in W$ and let $k = \frac{1}{|f|+1}$. Then $k \in L^\infty(\lambda)$, $k^{-1} \in L^\alpha$. It follows from Theorem 3.7 that there is a $s \in H^\infty(\lambda)$, $s^{-1} \in H^\alpha$ and an unimodular function u such that $k = us$, so $sf = \overline{u}ks = \overline{u} \frac{f}{|f|+1} \in L^\infty(\lambda)$. There is a sequence $\{s_n\}$ in $H^\infty(\lambda)$ such that $\alpha(s_n - s^{-1}) \rightarrow 0$. For each $n \in \mathbb{N}$, it follows from the lemma 3.8 that $s_n sf \in H^\infty(\lambda)H^\infty(\lambda)W \subset W$ and $s_n sf \in H^\infty(\lambda)L^\infty(\lambda) \subset L^\infty(\lambda)$, which implies that $s_n sf$ is a sequence in $W \cap L^\infty(\lambda)$, $\alpha(s_n sf - f) \leq \alpha(s_n - s^{-1})\|sf\|_\infty \rightarrow 0$. Thus $f \in \overline{W \cap L^\infty(\lambda)}^\alpha$. Therefore $W = \overline{W \cap L^\infty(\lambda)}^\alpha$. \square

COROLLARY 3.11. *A weak*-closed linear subspace M of $L^\infty(\lambda)$ satisfies $zM \subset M$ if and only if $M = \varphi H^\infty(\lambda)$ for some unimodular function φ or $M = \chi_E L^\infty(\lambda)$, for some Borel subset E of \mathbb{T} .*

PROOF. If $M = \varphi H^\infty(\lambda)$ for some unimodular function φ or $M = \chi_E L^\infty(\lambda)$, for some Borel subset E of \mathbb{T} , clearly, a weak*-closed linear subspace M of $L^\infty(\lambda)$ with $zM \subset M$. Conversely, since $zM \subset M$, and we have $z\overline{M}^{\|\cdot\|_{2,\lambda}} \subset \overline{M}^{\|\cdot\|_{2,\lambda}}$. Hence by Beurling-Helson-Lowdenslager theorem for $\|\cdot\|_{2,\lambda}$, we consider either $\overline{M}^{\|\cdot\|_{2,\lambda}} = \varphi H^2(\lambda)$ for some unimodular function φ , then $M = \overline{M}^{\|\cdot\|_{2,\lambda}} \cap L^\infty(\lambda) = \varphi H^2(\lambda) \cap L^\infty(\lambda)$; or $\overline{M}^{\|\cdot\|_{2,\lambda}} = \chi_E L^2(\lambda)$, for some Borel subset E of \mathbb{T} , in this case, $M = \overline{M}^{\|\cdot\|_{2,\lambda}} \cap L^\infty(\lambda) = \chi_E L^2(\lambda) \cap L^\infty(\lambda) = \chi_E L^\infty(\lambda)$, i.e., $M = \chi_E L^\infty(\lambda)$. \square

Now we obtain our main theorem, which extends the Chen-Beurling Helson-Lowdenslager Theorem.

THEOREM 3.12. *Suppose μ is Haar measure on \mathbb{T} and α is a continuous normalized gauge norm on $L^\infty(\mu)$. Suppose also that $c > 0$ and λ is a probability measure that is mutually absolutely continuous with respect to μ such that α is $c\|\cdot\|_{1,\lambda}$ -dominating and $\log|d\lambda/d\mu| \in L^1(\mu)$. Then a closed linear subspace W of $L^\alpha(\mu)$ satisfies $zW \subset W$ if and only if either $W = \varphi H^\alpha(\mu)$ for some unimodular function φ , or $W = \chi_E L^\alpha(\mu)$, for some Borel subset E of \mathbb{T} . If $0 \neq W \subset H^\alpha(\mu)$, then $W = \varphi H^\alpha(\mu)$ for some inner function φ .*

PROOF. Recall that $L^\infty(\mu) = L^\infty(\lambda)$, $L^\alpha(\mu) = L^\alpha(\lambda)$ and $H^\alpha(\mu) = H^\alpha(\lambda)$. The only if part is obvious. Let $M = W \cap L^\infty(\lambda)$, and in the theorem 2.2, we proved that there exists a measure λ such that $\lambda \ll \mu$ and $\mu \ll \lambda$ and there exists $c > 0$, $\forall f \in L^\infty(\mu) = L^\infty(\lambda)$, $\alpha(f) \geq c\|f\|_{1,\lambda}$. i.e., α is a continuous $c\|\cdot\|_{1,\lambda}$ -dominating normalized gauge norm on $L^\infty(\lambda)$. It follows from the (2) in theorem 3.10 that M is weak* closed in $L^\infty(\lambda)$. Since $zW \subset W$, it is easy to check that $zM \subset M$. Then by the corollary 3.11, we can conclude that either $M = \varphi H^\infty(\lambda)$ for some unimodular function φ or $M = \chi_E L^\infty(\lambda)$, for some Borel subset E of \mathbb{T} . By the (3) in theorem 3.10, if $M = \varphi H^\infty(\lambda)$, $W = \overline{W \cap L^\infty(\lambda)}^\alpha = \overline{M}^\alpha = \overline{\varphi H^\infty(\lambda)}^\alpha = \varphi H^\alpha(\mu)$, for some unimodular function φ . If $M = \chi_E L^\infty(\lambda)$, $W = \overline{W \cap L^\infty(\lambda)}^\alpha = \overline{M}^\alpha = \overline{\chi_E L^\infty(\lambda)}^\alpha = \chi_E L^\alpha = \chi_E L^\alpha(\mu)$, for some Borel subset E of \mathbb{T} . The proof is complete. \square

4. Which α 's have a good λ ?

In the preceding section we proved a version of Beurling's theorem for L^α when there is a probability measure λ on \mathbb{T} that is mutually absolutely continuous with respect to μ , such that α is $c\|\cdot\|_{1,\lambda}$ -dominating and $d\lambda/d\mu$ is log-integrable with respect to μ . How do we tell when such a good λ exists. Suppose ρ is a probability measure on \mathbb{T} that is mutually absolutely continuous with respect to μ such that

$$\int_{\mathbb{T}} \log(d\rho/d\mu) d\mu = -\infty.$$

Here are some useful examples.

EXAMPLE 4.1. Let $\alpha = \frac{1}{2}(\|\cdot\|_{1,\mu} + \|\cdot\|_{1,\rho})$. Then α is a continuous gauge norm. If we let $\lambda_1 = \rho$ and $\lambda_2 = \mu$ we see that $\alpha \geq \frac{1}{2}\lambda_k$ for $k = 1, 2$ and

$$\int_{\mathbb{T}} |\log(d\lambda_k/d\mu)| d\mu = \begin{cases} \infty & \text{if } k = 1 \\ 0 & \text{if } k = 2 \end{cases}.$$

Hence there is both a bad choice of λ and a good choice.

EXAMPLE 4.2. Suppose ρ is as in the preceding example and let $\alpha = \|\cdot\|_{1,\rho}$. Suppose λ is a probability measure that is mutually absolutely continuous with respect to μ and

$$\|\cdot\|_{1,\rho} = \alpha \geq c\|\cdot\|_{1,\lambda} \text{ for some constant } c.$$

It follows that $d\lambda/d\rho \leq c$ a.e., and thus

$$\int_{\mathbb{T}} \log(d\lambda/d\mu) d\mu = \int_{\mathbb{T}} \log(d\lambda/d\rho) d\mu + \int_{\mathbb{T}} \log(d\rho/d\mu) d\mu \leq \log \varepsilon + (-\infty) = -\infty.$$

In this case there is no good λ .

5. A special case.

Suppose λ is any probability measure that is mutually absolutely continuous with respect to μ and $\alpha = \|\cdot\|_{p,\lambda}$ for some p with $1 \leq p < \infty$. Assume λ is bad, i.e., $\int_{\mathbb{T}} \left| \log \frac{d\lambda}{d\mu} \right| d\mu = \infty$. In this case, we define a bijective isometry mapping $U : L^p(\lambda) \rightarrow L^p(\mu)$ by $Uf = g^{\frac{1}{p}}f$. Let $H^p(\lambda)$ be the α -closure of all polynomials, then $H^p(\lambda)$ is a closed subspace of $L^p(\lambda)$ and $zH^p(\lambda) \subseteq H^p(\lambda)$. Therefore, $g^{\frac{1}{p}}H^p(\lambda)$ is a z -invariant closed subspace of $L^p(\mu)$. By Beurling-Helson-Lowdenslager theorem, we have

$$g^{\frac{1}{p}}H^p(\lambda) = \chi_E L^p(\mu) \text{ for some Borel set } E \subseteq \mathbb{T}, \text{ or } \varphi H^p(\mu), \text{ where } |\varphi| = 1.$$

If $g^{\frac{1}{p}}H^p(\lambda) = \chi_E L^p(\mu)$, then $H^p(\lambda) = L^p(\lambda)$, in this case, $\varphi H^p(\lambda) = \varphi L^p(\lambda)$, where $|\varphi| = 1$. If $M_0 = \frac{1}{g^{1/p}}H^p(\mu)$, then M_0 is a proper z -invariant closed subspace of $L^p(\lambda)$, and $M_0 \neq \chi_E L^p(\lambda)$. Therefore, Beurling-Helson-Lowdenslager theorem is not true for this case. However, we have the following theorem

THEOREM 5.1. Suppose λ is any probability measure that is mutually absolutely continuous with respect to μ and $\alpha = \|\cdot\|_{p,\lambda}$ for some p with $1 \leq p < \infty$. Also assume $\int_{\mathbb{T}} \left| \log \frac{d\lambda}{d\mu} \right| d\mu = \infty$. If M is a closed subspace of $L^\alpha(\lambda)$, then $zM \subseteq M$ if and only if

- (1) $M = \varphi M_0$ for some unimodular function φ , where $M_0 = \frac{1}{g^{1/p}} H^p(\mu)$, or
- (2) $M = \chi_E L^\alpha(\lambda)$ for some Borel subset E of \mathbb{T} .

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